The Envelope Theorem, Euler and Bellman Equations, without Differentiability*

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Abstract

We extend the envelope theorem, the Euler equation, and the Bellman equation to dynamic constrained optimization problems where binding constraints can give rise to non-differentiable value functions and multiplicity of Lagrange multipliers. The envelope theorem – an extension of Milgrom and Segal's (2002) theorem – establishes a relation between the Euler and the Bellman equation. We show that solutions and multipliers of the Bellman equation may fail to satisfy the respective Euler equations, in contrast with solutions and multipliers of the infinite-horizon problem. In standard dynamic optimisation problems the failure of Euler equations results in *inconsistent multipliers*, but not in non-optimal outcomes. However, in problems with *forward-looking* constraints this failure can result in inconsistent promises and non-optimal outcomes. We also show how the inconsistency problem can be resolved by an envelope selection condition and a minimal extension of the co-state. We extend the theory of recursive contracts of Marcet and Marimon (1998, 2017) to the case where the value function is non-differentiable, resolving a problem pointed out in Messner and Pavoni (2004).

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1 Introduction

The Euler equation and the Bellman equation are the two basic tools used to analyse dynamic optimisation problems. Euler equations are the first-order inter-temporal *necessary conditions* for optimal solutions and, under standard concavity-convexity assumptions, they are also *sufficient conditions*, provided that a transversality condition holds. Euler equations are usually second-order difference equations. The Bellman equation allows the transformation of an infinite-horizon optimisation problem into a recursive problem, resulting in time-independent policy functions determining the actions as functions of the states. The envelope theorem provides the bridge between the Bellman equation and the Euler equations, confirming the necessity of the latter for the former. The envelope theorem allows us to reduce the second-order difference equation system of Euler equations to a first-order system, fully determined by the policy function of the Bellman equation with corresponding initial conditions, provided that the value function is differentiable.

Differentiability makes the bridge between the Bellman equation and the Euler equation tight. If the value function is differentiable, the state provides univocal information about the derivative and, therefore, the inter-temporal change of values across states (Bellman) is uniquely associated with the change of marginal values (Euler) via the envelope theorem – as a result, the envelope theorem allows for the passage of properties between the Euler and the Bellman equations. That is, the necessity and sufficiency properties of the Euler equations on the one hand, and the properties of Bellman policy functions on the other. However, the value function may not be differentiable when constraints are binding and, in this case, knowing the state and its value does not provide univocal information about the derivative. Sub-differential calculus (e.g. Rockafellar (1970, 1981)) comes into play, but needs to be properly developed in order to characterise the envelope bridge between the Euler and Bellman equations, without differentiability. This is the objective of this paper.

Recursive methods have been widely applied in macroeconomics over the last 30 years since the publication of Stokey et al. (1989), using the standard framework where assumptions, such as interiority of optimal paths, imply the differentiability of the value function. However, the differentiability issue cannot be ignored in a wide range of current applications. Models where households, firms, or countries, may face binding constraints in equilibrium are, nowadays, more the norm than the exception. Furthermore, contractual models often have *forward-looking constraints* (i.e. involving future equilibrium outcomes). It is well known that optimization problems with forward-looking constraints may have time-inconsistent solutions. That is, re-optimising in an infinite-horizon problem at date t with initial value given as x_t^* – where x_t^* is the date-t state of the optimal path from date 0 – may lead to a solution which is not part of an optimal solution from date 0. Standard dynamic programming fails, but as Marcet and Marimon (2017) have shown, the *saddle-point Bellman equation* with an extended co-state can be used to recover recursive structure of the problem. Nevertheless, the differentiability problem caused by binding constraints remains and, as we show, it is more perverse when constraints are *forward-looking*. Therefore, we focus our analysis on differentiability problems arising from binding constraints.

Our analysis covers the trilogy already mentioned. First, we study the envelope theorem for static constrained optimisation problems without assuming differentiability of the value function or interiority of the solutions. We extend the envelope theorem for directional derivatives of Milgrom and Segal (2002, Corollary 5)[1] by relaxing some of their assumptions, and provide characterizations of the superdifferential of a concave value function and the subdifferential of a convex value function. From the envelope theorem, we derive several sufficient conditions for differentiability of the value function. For example, if there is a unique saddle-point, the value function is differentiable and the standard form of the envelope theorem holds. A sufficient condition for differentiability of concave value function is that the saddle-point multiplier be unique. For convex value function, a sufficient condition for differentiability is that the solution be unique. We provide examples of applications of our results to static optimization problems. This first part is covered in Sections 2 and 3.

Second, we turn to *the Euler equations* of the dynamic optimisation problem. Unless the solution is interior, marginal values of the constraints (i.e. the Lagrange multipliers) are part of the Euler equations. Applying the envelope theorem of Section 3, we show how the Euler equations can be derived from the Bellman equation without assuming differentiability of the value function. If there are multiple multipliers of the Bellman equation, then some sequences of multipliers may fail to satisfy the Euler equations and not be saddle-point multipliers of the infinite-horizon

[1]Earlier contributions include Dubeau and Govin (1982), Rockafellar (1984), Bonnisseau and Le Van (1996), and references listed in Milgrom and Segal (2002). An extension to non-smooth optimisation problems has recently been provided in Morand, Reffett and Tarafdar (2015).

problem. In particular, restarting the Bellman equation at a point of non-differentiability of the value function may result in *time-inconsistent multipliers*, which do not satisfy the Euler equations. We introduce an envelope selection condition that guarantees that multipliers generated from the Bellman equation satisfy the Euler equations. The envelope selection is a consistency condition on multipliers and does not affect solutions in standard dynamic optimization problems without forward-looking constraints. The recursive method of solving dynamic programming problems can be extended to provide solutions with consistent multipliers by expanding the co-state to include a subgradient of the value function. If the value function is differentiable, this co-state is redundant, since the subgradient is unique. Extending the well-known result of Benveniste and Scheinkman (1979), we show that the concave value function is differentiable if the multiplier of the Bellman equation is unique.[2] Section 4 contains this analysis for standard dynamic optimization problems, and an example.

Third, in Section 5, we further develop Marcet and Marimon's (1998, 2017) *saddle-point method* of solving dynamic optimisation problems with forward-looking constraints. The problem of inconsistent multipliers in the absence of forward-looking constraints becomes inconsistency of solutions and multipliers in the presence of such constraints. We show that the envelope selection condition guarantees that solutions and multipliers generated from the saddle-point Bellman equation satisfy the Euler equations without assuming that the value function is differentiable. Furthermore, we show that the envelope selection condition is equivalent to the *intertemporal consistency condition* introduced by Marcet and Marimon, motivated by an example of Messner and Pavoni (2004) showing that the saddle-point Bellman equation with a non-differentiable value function can generate non-optimal outcomes in recursive contracts. Imposing the envelope selection condition in this example (see Example 4 in Section 5) results in a recursive optimal solution satisfying the Euler equations.

Although the Euler equation is part of the standard 'toolkit' of dynamic optimization problems (e.g. Stokey et al. (1989)), we are not aware of any discussion of the consistency problem presented in this paper. This is possibly due to the fact that most of the analyses, and computations, using Euler equations implicitly assume that the value function is differentiable.[3] This

[2]Rincón-Zapatero and Santos (2009) study differentiability of concave value function in dynamic optimization problems assuming that a constrained qualification condition holds.

[3]Cole and Kubler (2012) identify the consistency problem of the saddle-point method with forward-looking

paper provides the the necessary ingredients for extending the existing computational methods for solving this broader class of models. Section 6 provides further discussion and conclusions.

2 The Envelope Theorem

We consider the following parametric constrained optimization problem:

$$\max_{y \in Y} f(x, y) \tag{1}$$

subject to

$$h_1(x, y) \geq 0, \ \ldots, \ h_k(x, y) \geq 0. \tag{2}$$

Parameter x lies in the set $X \subset \Re^m$. Choice variable y lies in $Y \subset \Re^n$. Objective function f is a real-valued function on $Y \times X$. Each constraint function h_i is a real-valued function on $Y \times X$.[4] The value function of the problem (1–2) is denoted by $V(x)$.

The Lagrangian function associated with (1–2) is

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda h(x, y), \tag{3}$$

where $\lambda \in \Re^k_+$ is a vector of (positive) multipliers[5]. It is well known that if (y^*, λ^*) is a saddle-point of $\mathcal{L}$, that is, if

$$\mathcal{L}(x, y, \lambda^*) \leq \mathcal{L}(x, y^*, \lambda^*) \leq \mathcal{L}(x, y^*, \lambda), \tag{4}$$

for every $y \in Y$ and $\lambda \in \Re^k_+$, then y^* is a solution to (1–2). Further, the slackness condition, $\lambda^*_i h_i(x, y^*) = 0$, holds for every i and consequently

$$V(x) = \mathcal{L}(x, y^*, \lambda^*) = \text{SP} \ \min_{\lambda \geq 0} \max_{y \in Y} \mathcal{L}(x, y, \lambda), \tag{5}$$

where SP denotes the *saddle-point* operator defined by (4) with minimization over λ and maximization over y.[6] The set of saddle-points of $\mathcal{L}$ at x is a product of two sets and is denoted by

constraints, but offer a way to resolve it within a limited class of models (e.g., partnerships with two agents). There is no use of the envelope theorem in their approach.

[4]Note that optimization problems with equality constraints can be represented in form (1–2) by taking $h_i = -h_j$ for some i and j.

[5]We use the product notation: $\lambda h(x, y) = \sum_{i=1}^k \lambda_i h_i(x, y)$.

[6]This SP notation was introduced in Marcet and Marimon (2017). The $\min$ ($\max$) operator only denotes which variables are being minimised (maximised) in the saddle-point.

$Y^*(x) \times \Lambda^*(x)$ where $Y^*(x) \subset Y$ and $\Lambda^*(x) \subset \Re^k_+$, see Rockafellar (1970), Lemma 36.2. If (y^*, λ^*) is a saddle-point of $\mathcal{L}$, y^* will be called a saddle-point solution and λ^* a saddle-point multiplier. The slackness condition implies that if the ith constraint is not binding, that is, $h_i(x, y^*) > 0$ for a saddle-point solution y^*, then $\lambda_i^* = 0$ for every saddle-point multiplier λ^*.

We shall impose the following conditions:

A1. Y is convex and compact.

A2. f and h_i are continuous functions of (x, y), for every i.

A3. For every $x \in X$ and every i, there exists $\hat{y}_i \in Y$ such that $h_i(x, \hat{y}_i) > 0$ and $h_j(x, \hat{y}_i) \geq 0$ for $j \neq i$.

A4. $Y^*(x) \times \Lambda^*(x) \neq \emptyset$ for every $x \in X$.

Assumptions A1-2 are standard. Assumption A3 essentially says that none of the inequality constraints $h_i(x, y) \geq 0$ alone can be replaced by equality constraint $h_i(x, y) = 0$. It is weaker than Slater's condition which requires that there is $\bar{y} \in Y$ such that $h_i(x, \bar{y}) > 0$ for every i. If all functions h_i are concave in y, then A3 is equivalent to Slater's condition. Assumption A4 says that the set of saddle-points of $\mathcal{L}$ is nonempty for every x. It holds if functions f and h_i are concave in y and the Slater's condition holds.

The set of saddle-point solutions $Y^*(x)$ is a subset of the set of solutions to (1–2). These two sets are equal if functions f and h_i are concave in y and the Slater's condition holds. If f and h_i are differentiable in y, then the Kuhn-Tucker first-order conditions hold for every saddle-point of $\mathcal{L}$ and the set of saddle-point multipliers $\Lambda^*(x)$ is a subset of the set of Kuhn-Tucker multipliers. Those two sets of multipliers are equal if functions f and h_i are differentiable and concave in y.

The envelope theorem is best stated in terms of directional derivatives. We first consider one-dimensional parameter set X – a convex subset of the real line. Directional derivatives are then the left- and right-hand derivatives. The right- and left-hand derivatives of the value function V at x are

$$V'(x+) = \lim_{t \to 0+} \frac{V(x+t) - V(x)}{t}, \tag{6}$$

and

$$V'(x-) = \lim_{t \to 0-} \frac{V(x+t) - V(x)}{t}, \tag{7}$$

if the limits exist.

We have the following result:

Theorem 1: Suppose that $X \subset \Re$, conditions A1-A4 hold, and partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial h_i}{\partial x}$ are continuous functions of (x, y). Then the value function V is right- and left-hand differentiable at every $x \in intX$ and the directional derivatives are

$$V'(x+) = \max_{y^* \in Y^*(x)} \min_{\lambda^* \in \Lambda^*(x)} \big[\frac{\partial f}{\partial x}(x, y^*) + \lambda^* \frac{\partial h}{\partial x}(x, y^*)\big] \quad (8)$$

and

$$V'(x-) = \min_{y^* \in Y^*(x)} \max_{\lambda^* \in \Lambda^*(x)} \big[\frac{\partial f}{\partial x}(x, y^*) + \lambda^* \frac{\partial h}{\partial x}(x, y^*)\big], \quad (9)$$

where the order of maximum and minimum does not matter.

Theorem 1 is an extension of Corollary 5 in Milgrom and Segal (2002). It is worth pointing out that differentiability of functions f and h_i with respect to the variable y is not assumed in Theorem 1.[7] This will be important in applications to dynamic programming in Sections 4 and 5.

For a multi-dimensional parameter set X in $\Re^m$, the directional derivative of the value function V at $x \in X$ in the direction $\hat{x} \in \Re^m$ such that $x + \hat{x} \in X$ is defined as

$$V'(x; \hat{x}) = \lim_{t \to 0+} \frac{V(x + t\hat{x}) - V(x)}{t}.$$

If partial derivatives of V exist, then the directional derivative $V'(x; \hat{x})$ is equal to the scalar product $DV(x)\hat{x}$, where $DV(x)$ is the vector of partial derivatives, i.e. the gradient vector.

Theorem 1 can be applied to the single-variable value function $\tilde{V}(t) \equiv V(x + t\hat{x})$ for which it holds $\tilde{V}'(0+) = V'(x; \hat{x})$. If $D_x f(x, y)$ and $D_x h_i(x, y)$ are continuous functions of (x, y), then it follows that the directional derivative of V is

$$V'(x; \hat{x}) = \max_{y^* \in Y^*(x)} \min_{\lambda^* \in \Lambda^*(x)} \left[D_x f(x, y^*) + \lambda^* D_x h(x, y^*)\right] \hat{x}, \quad (10)$$

and the order of maximum and minimum does not matter.

[7]Dubeau and Govin (1982) and Rockafellar (1984) assume differentiability of functions f and h_i and use Kuhn-Tucker multipliers instead of saddle-point multipliers. Most of their results provide bounds on directional derivatives of V.

3 Differentiability and Subdifferentials of the Value Function

3.1 Differentiability

The value function V on $X \subset \Re$ is differentiable at x if the one-sided derivatives are equal to each other. Sufficient conditions for differentiability can be obtained from Theorem 1.

Corollary 1: Under the assumptions of Theorem 1, each of the following conditions is sufficient for differentiability of value function V at $x \in intX$:

(i) there is a unique saddle-point,

(ii) there is a unique saddle-point solution and h_i does not depend on x for every i.

(iii) there is a saddle-point solution with non-binding constraints, and $\frac{\partial f}{\partial x}$ does not depend on y.[8]

(iv) there is a unique saddle-point multiplier and $\frac{\partial f}{\partial x}$ and $\frac{\partial h_i}{\partial x}$ do not depend on y, for every i.

Condition (i) holds if there is a unique saddle-point solution with non-binding constraints so that zero is the unique saddle-point multiplier. The condition of $\frac{\partial f}{\partial x}$ or $\frac{\partial h_i}{\partial x}$ not depending on y in part (iv) is essentially the additive separability of f and h_i in x and y. Under the separability condition, uniqueness of multiplier is necessary and sufficient for differentiability of the value function. A result related to Corollary 1 (iii) can be found in Kim (1993).

A sufficient condition for uniqueness of saddle-point solution to (1–2) is that f be strictly concave and h_i be concave in y. A sufficient condition for uniqueness of saddle-point multiplier is the following standard Constrained Qualification condition which implies uniqueness of the Kuhn-Tucker multiplier:

CQ (1) f and h_i are continuously differentiable functions of y,

(2) vectors $D_y h_i(x, y^*)$ for $i \in I(x, y^*)$ are linearly independent, where $I(x, y^*) = \{i : h_i(x, y^*) = 0\}$ is the set of binding constraints.

[8] It is sufficient that the constraints with h_i depending on x are non-binding. Other constraints may bind.

A weaker form of Constrained Qualification which is necessary and sufficient for uniqueness of Kuhn-Tucker multiplier can be found in Kyparisis (1985). Note that CQ holds vacuously for solution y^* with non-binding constraints.

Under condition (i) or (iv) of Corollary 1, the derivative of the value function is

$$V'(x) = \frac{\partial f}{\partial x}(x, y^*) + \lambda^* \frac{\partial h}{\partial x}(x, y^*). \tag{11}$$

Under condition (ii) or (iii), it holds

$$V'(x) = \frac{\partial f}{\partial x}(x, y^*). \tag{12}$$

For the multi-dimensional parameter set X in $\Re^m$, the value function is differentiable if $V'(x; \hat{x}) = -V'(x; -\hat{x})$ for every $\hat{x} \in \Re^m$. This holds under any of the sufficient conditions of Corollary 1 with $D_x f$ and $D_x h_i$ substituted for partial derivatives in (iv) and (v). If V is differentiable, then the gradient $DV(x)$ is well defined, and the multi-dimensional counterpart of (11) is

$$DV(x) = D_x f(x, y^*) + \lambda^* D_x h(x, y^*). \tag{13}$$

Results of this section and Section 2 can be extended to minimization problems and saddle-point problems. We present an extension to saddle-point problems in Appendix A.

3.2 Concave and Convex Value Functions

If the value function is concave or convex, the envelope theorem can be stated using the superdifferential or the subdifferential, respectively, for a multi-dimensional parameter set. We consider the concave case first.

Sufficient conditions for V to be concave are stated in the following well-known result, the proof of which is omitted.

Proposition 1. If the objective function f and all constraint functions h_i are concave functions of (x, y) on $Y \times X$, then the value function V is concave.

The superdifferential $\partial V(x)$ of the concave value function V is the set of all vectors $\phi \in \Re^m$ such that

$$V(x') + \phi(x - x') \leq V(x) \quad \text{for every} x' \in X.$$

We have the following:

Theorem 2: Suppose that conditions A1-A4 hold, derivatives $D_x f$ and $D_x h_i$ are continuous functions of (x, y) for every i, and V is concave. Then

$$\partial V(x) = \bigcap_{y^* \in Y^*(x)} \bigcup_{\lambda^* \in \Lambda^*(x)} \{D_x f(x, y^*) + \lambda^* D_x h(x, y^*)\} \tag{14}$$

for every $x \in intX$.

Sufficient conditions for differentiability of concave value function follow from Theorem 2.

Corollary 2: Under the assumptions of Theorem 2, the following hold for $x \in intX$:

(i) If the saddle-point multiplier is unique, then value function V is differentiable at x and (13) holds for every $y^* \in Y^*(x)$.

(ii) If h_i does not depend on x for every i, then value function V is differentiable at x and (13) holds for every $y^* \in Y^*(x)$.

In Corollary 2 (i), it is sufficient that the multiplier is unique for the constraints with h_i depending on x. Corollary 2 (i) implies that the value function is differentiable if there is a solution with non-binding constraints - those that depend on x - for then the unique saddle-point multiplier is zero. A saddle-point multiplier may be unique even if some constraints are binding. Examples are given in Section 3.3. Corollary 2 (ii) extends Corollary 3 in Milgrom and Segal (2002) to parametrized constraints.

We now provide a similar characterization for convex value functions. Sufficient conditions for V to be convex are stated without proof in the following:

Proposition 2. If the objective function $f(y, \cdot)$ is convex in x for every $y \in Y$ and all constraint functions h_i are independent of x, then the value function V is convex.

If V is convex, then the subdifferential $\partial V(x)$ is the set of all vectors $\phi \in \Re^m$ such that[9]

$$V(x') + \phi(x - x') \geq V(x) \quad for\ every\ x' \in X.$$

We have the following:

[9]We use the same notation for the superdifferential and the subdifferential as is customary in the literature.

Theorem 3: Suppose that conditions A1-A3 hold, derivatives $D_x f$ and $D_x h_i$ are continuous functions of (x, y) for every i, and V is convex[10]. Then

$$\partial V(x) = \bigcap_{\lambda^* \in \Lambda^*(x)} co\Big(\bigcup_{y^* \in Y^*(x)} \{D_x f(x, y^*) + \lambda^* D_x h(x, y^*)\}\Big), \tag{15}$$

for every $x \in intX$, where co( ) denotes the convex hull.

Sufficient conditions for differentiability of convex value function follow from Theorem 3.

Corollary 3: Under the assumptions of Theorem 3, if the saddle-point solution is unique at $x \in intX$, then value function V is differentiable and (13) holds for every $\lambda^* \in \Lambda^*(x)$.

3.3 Examples

Example 1 (Perturbation of constraints): suppose that the objective function f in (1) is independent of the parameter x and constraint functions are of the form $h_i(x, y) = \hat{h}_i(y) - x_i$. This optimization problem is a perturbation of the non-parametric problem with objective function f and constraint functions $\hat{h}_i$. Rockafellar (1970) provides an extensive discussion of the concave perturbed problem.

Corollary 1 (iv) implies that if the saddle-point multiplier λ^* is unique, then the value function is differentiable and $DV(x) = -\lambda^*$ by (13). (See 29.1.3 in Rockafellar (1970) for the concave perturbed problem.) If f and $\hat{h}^i$ are concave for every i, then V is concave and the superdifferential of V is $\partial V(x) = -\Lambda^*(x)$.

Example 2 (A planner's problem): consider the resource allocation problem in an economy with k agents. The planner's problem is

$$\max_{\{c_i\}} \sum_{i=1}^{k} \mu_i u_i(c_i) \tag{16}$$

$$\text{s.t.} \sum_{i=1}^{n} c_i \le x, \tag{17}$$

$$c_i \ge 0, \quad \forall i,$$

[10] Oyama and Takenawa (2017) provides an example showing that the continuity of the derivatives $D_x f$ and $D_x h_i$ is a necessary assumption.

where $\mu = (\mu_1, \ldots, \mu_k) \in \Re^k_{++}$ is a vector of welfare weights and $x \in \Re^L_+$ represents total resources. Utility functions u_i are continuous and increasing. Let $V(x, \mu)$ be the value of (16) as function of weights μ and total resources x. It follows from Corollary 1 (iv) that V is differentiable in x if the saddle-point multiplier of constraint (17) is unique. If utility functions u_i are differentiable, then the CQ condition holds, implying that the multiplier is unique. The derivative is $D_x V = \lambda^*$, where λ^* is the multiplier of the constraint (17). V is a convex function of μ. The subdifferential $\partial_\mu V$ is (by Theorem 3) the convex hull of the set of vectors $(u_1(c_1^*), \ldots, u_k(c_k^*))$ over all saddle-point solutions c^* to (16). V is differentiable in μ if the saddle-point solution is unique.

Consider an example with $L = 1, k = 2$, and $u_i(c) = c$. Let the welfare weights be parametrized by a single parameter μ so that $\mu_1 = \mu$ and $\mu_2 = 1 - \mu$ with $0 < \mu < 1$. The value function is $V(x, \mu) = \max\{\mu, 1-\mu\}x$. It is differentiable with respect to μ at every $\mu \neq \frac{1}{2}$ and every x. The solution c^* is unique for every $\mu \neq \frac{1}{2}$. V is not differentiable with respect to μ at $\mu = \frac{1}{2}$. The left-hand directional derivative at $\mu = \frac{1}{2}$ is $-x$ while the right-hand derivative is x in accordance with Theorem 1. V is everywhere differentiable with respect to x.

4 Dynamic Optimization and Euler Equations

In this section we extend the standard results of dynamic programming to non-differentiable value functions. Using the results of Sections 2 and 3, we show how to derive Euler equations from the Bellman equation without differentiability of the value function. If the value function is non-differentiable, then there may be sequences of solutions and multipliers generated from the Bellman equation for which Euler equations do not hold because multipliers are inconsistent. We develop a recursive method of selecting solutions with consistent multipliers.

We consider the following dynamic constrained maximization problem studied in Stokey et al. (1989):

$$\max_{\{x_t\}_{t=1}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \tag{18}$$
$$\text{s.t. } h_i(x_t, x_{t+1}) \geq 0, \;\; i = 1, ..., k, \; t \geq 0,$$

for given $x_0 \in X$, where $\{x_t\}_{t=1}^{\infty}$ is a bounded sequence (i.e., $\{x_t\} \in \ell^n_\infty$) such that $x_t \in X \subset \Re^n$

for every t. Functions F and h_i are real-valued functions on $X \times X$. We impose the following conditions:

D1. X is convex.

D2. There exists $\{\hat{x}_t\} \in \ell^n_\infty$ such that $h_i(\hat{x}_t, \hat{x}_{t+1}) > 0$ for every $t \geq 0$ and every i.

D3. F and h_i are bounded, and $\beta \in (0, 1)$.

D4. F and h_i are concave functions of (x, y) on $X \times X$,

D5. F and h_i are increasing and differentiable on X.

The saddle-point problem associated with (18) is

$$\text{SP} \max_{\{x_t\}_{t=1}^\infty} \min_{\{\lambda_t\}_{t=1}^\infty, \lambda_t \geq 0} \sum_{t=0}^\infty \beta^t \big[F(x_t, x_{t+1}) + \lambda_{t+1} h(x_t, x_{t+1})\big], \tag{19}$$

for given $x_0 \in X$, where $\lambda_t \in \Re^k$ are Lagrange multipliers and $\{x_t\} \in \ell^n_\infty$. It follows from Dechert (1992) that if a sequence $\{x^*_t\}$ is a solution to (18), then under assumptions D1-D4 there exists a summable sequence of multipliers $\{\lambda^*_t\} \in \ell^k_1$ such that $\{x^*_t, \lambda^*_t\}$ is a saddle-point of (19). Conversely, if $\{x^*_t, \lambda^*_t\}$ is a saddle-point of (19), then $\{x^*_t\}$ is a solution to (18).

By a standard variational argument, the first-order *necessary conditions* for saddle-point $\{x^*_t, \lambda^*_t\}_{t=1}^\infty$ of (19) are the following *intertemporal Euler equations*:

$$D_y F(x^*_t, x^*_{t+1}) + \lambda^*_{t+1} D_y h(x^*_t, x^*_{t+1}) + \beta\big[D_x F(x^*_{t+1}, x^*_{t+2}) + \lambda^*_{t+2} D_x h(x^*_{t+1}, x^*_{t+2})\big] = 0 \tag{20}$$

for every $t \geq 0$. Equations (20) together with complementary slackness conditions and the given constraints, define a system of second-order difference equations for $\{x^*_t, \lambda^*_t\}$ with $x^*_0 = x_0$. Complementary slackness conditions are

$$\lambda^*_{t+1} h(x^*_t, x^*_{t+1}) = 0, \;\; h(x^*_t, x^*_{t+1}) \geq 0. \tag{21}$$

The sufficiency of the Euler equation and a transversality condition, see Stokey *et al.* (1989), continue to hold when a constraint is binding.

Proposition 3: *Suppose that conditions D1-D5 hold. Let* $\{x_t^*, \lambda_t^*\}$*, with* $\{x_t^*\} \in \ell_\infty^n$*,* $x_0^* = x_0$*,* $\lambda_t^* \geq 0$*, and* $h(x_t^*, x_{t+1}^*) \geq 0$ *for every* t*, satisfy the Euler equations (20) and the complementary slackness conditions (21). If the transversality condition*

$$\lim_{t \to \infty} \beta^t \left[D_x F(x_t^*, x_{t+1}^*) + \lambda_{t+1}^* D_x h(x_t^*, x_{t+1}^*) \right] = 0, \tag{22}$$

holds, then $\{x_t^*, \lambda_t^*\}$ *is a saddle-point of (19). In particular,* $\{x_t^*\}$ *is a solution to (18).*

Proof: see Appendix.

Let $V(x_0)$ be the value function of (18). The value function satisfies the Bellman equation

$$\begin{aligned} V(x) = \max_y \quad & \{F(x, y) + \beta V(y)\} \\ \text{s.t.} \quad & h_i(x, y) \geq 0, \;\; i = 1, ..., k \end{aligned} \tag{23}$$

The value function V is concave and bounded under assumptions D1-4. If $D_x F$ and $D_x h_i$ are continuous functions of (x, y) for every i, then by envelope Theorem 2, the superdifferential of V is

$$\partial V(x) = \bigcap_{y^* \in Y^*(x)} \bigcup_{\lambda^* \in \Lambda^*(x)} \{D_x F(x, y^*) + \lambda^* D_x h(x, y^*)\} \tag{24}$$

where $Y^*(x)$ is the set of saddle-point solutions and $\Lambda^*(x)$ is the set of saddle-point multipliers at x. Corollary 2 (i) implies that V is differentiable if saddle-point multiplier λ^* is unique. If there is a solution y^* with non-binding constraints, then the unique multiplier is zero and V is differentiable at x. This is the well-known result of Benveniste and Scheinkman (1979). Note also that if V is differentiable at y^* and the Constraint Qualification condition holds, then the saddle-point multiplier is unique and V is differentiable at x.

For every solution $\{x_t^*\}$ to (18), x_{t+1}^* is a solution to the Bellman equation (23) at x_t^* for every $t \geq 0$. The converse holds as well under assumptions D1- D3, see Stokey *et al.* (1989; Theorem 4.3). The latter result not only shows that it is sufficient to sequentially solve the Bellman equation (23) to obtain a solution to (18) but also that solutions are *time-consistent*. That is, if $\{x_t^*\}_{t=1}^\infty$ is a solution to (18) at x_0 and the Bellman equation is restarted at x_τ^* the resulting solution – say, $\{x_t^*\}_{t=1}^\tau, \{\widehat{x}_t\}_{t=\tau+1}^\infty$ – is also a solution to (18) at x_0.

It is well-known that if constraints are not binding and the value function is differentiable, then the first-order conditions of the Bellman equation together with the envelope theorem imply the Euler equations. Indeed, (24) simplifies then to $DV(x) = D_xF(x, y^*)$, the Euler equation simplifies to

$$D_yF(x_t^*, x_{t+1}^*) + \beta D_xF(x_{t+1}^*, x_{t+2}^*) = 0,$$

and the latter obtains from the first-order conditions of (23). In general, it is convenient to introduce the *saddle-point Bellman equation* corresponding to (23):

$$V(x) = \mathrm{SP} \min_{\lambda \geq 0} \max_{y} \left\{ F(x,y) + \lambda h(x,y) + \beta V(y) \right\}. \tag{25}$$

The set of saddle-points of (25) is the product set $Y^*(x) \times \Lambda^*(x)$. The first-order condition for saddle-point (y^*, λ^*) of (25) states that there exists a subgradient vector $\phi^* \in \partial V(y^*)$ such that

$$D_yF(x, y^*) + \lambda^* D_yh(x, y^*) + \beta\phi^* = 0, \tag{26}$$

see Rockafellar (1981, Ch.5).

If $\{x_t^*, \lambda_t^*\}$ is a saddle-point of (19), then $(x_{t+1}^*, \lambda_{t+1}^*)$ is a saddle-point of the *saddle-point Bellman equation* (25) at x_t^* for every $t \geq 0$. The converse implication requires a consistency condition that involves subgradients $\{\phi_t^*\}$ obtained from the first-order conditions for $\{x_t^*, \lambda_t^*\}$.

Proposition 4: *Suppose that conditions D1-D5 hold and D_xF and D_xh_i are continuous functions for every i. Let $\{x_t^*, \lambda_t^*\}_{t=1}^{\infty}$, with $\{x_t^*\} \in \ell_\infty^n$, be a sequence of saddle-points generated by the saddle-point Bellman equation (25), starting at $x_0^* = x_0$, and let $\{\phi_t^*\}_{t=0}^{\infty}$ be the corresponding sequence of subgradients satisfying (26), with $\phi_t^* \in \partial V(x_t^*)$ for all t. If the following envelope selection condition*

$$\phi_t^* = D_xF(x_t^*, x_{t+1}^*) + \lambda_{t+1}^* D_xh(x_t^*, x_{t+1}^*) \tag{27}$$

holds for every $t \geq 0$, then $\{x_t^*, \lambda_t^*\}_{t=1}^{\infty}$ is a saddle-point of (19).

Proof: The first-order condition (26) for $(x_{t+1}^*, \lambda_{t+1}^*)$ at x_t^*, for $t \geq 1$, is

$$D_yF(x_t^*, x_{t+1}^*) + \lambda_{t+1}^* D_yh(x_t^*, x_{t+1}^*) + \beta\phi_{t+1}^* = 0. \tag{28}$$

Eq. (28) together with the envelope selection condition (27) for ϕ_{t+1}^* imply that the Euler equation (20) holds at x_t^*. The transversality condition (22) can be written as $\lim_{t\to\infty} \beta^t \phi_t^* = 0$. Since the sequence $\{x_t^*\}$ is bounded, subgradients $\{\phi_t^*\}$ are bounded, too (see Rockafellar (1970), Theorem 24.7). This implies (22). The conclusion follows now from Proposition 3. □

The *envelope selection condition* (27) guarantees *consistency* of multipliers generated by the saddle-point Bellman equation. It can be dispensed with if the saddle-point multiplier is unique (which is sufficient for value function V to be differentiable) but not if there are multiple multipliers. Inconsistency of multipliers occurs if, given (x_t^*, λ_t^*) and $\phi_t^* \in \partial V(x_t^*)$ satisfying the first-order condition (28) at x_{t-1}^*, a multiplier λ_{t+1}^* is chosen without satisfying the envelope selection condition (27) when solving (28) at x_t^*. Then the Euler equation (20) is not satisfied for λ_t^* and λ_{t+1}^*, and the sequence $\{x_t^*, \lambda_t^*\}_{t=1}^{\infty}$ need not be a saddle-point of (19). This may happen if V is not differentiable at x_t^* and the saddle-point Bellman equation (25) is restarted at x_t^* without recalling the selection $\phi_t^* \in \partial V(x_t^*)$ previously made.

Proposition 4 does not provide a recursive method of generating consistent solutions from the saddle-point Bellman equation, but it clearly suggests what that method should be: extend the state x with a co-state $\phi \in \partial V(x)$, and impose the envelope selection condition. To that end, we define the *selective value function* $V^s(x;\phi)$ of state x and *co-state* $\phi \in \partial V(x)$ as the value function of the saddle-point Bellman equation (25) with the additional restriction that the saddle-point satisfies the envelope selection condition. That is,

$$\begin{aligned} V^s(x;\phi) \quad &= \quad \text{SP} \min_{\lambda \geq 0} \max_{y} \{F(x,y) + \lambda h(x,y) + \beta V(y)\} \\ \text{s.t.} \qquad & \qquad D_x F(x,y) + \lambda D_x h(x,y) = \phi. \end{aligned} \tag{29}$$

Clearly, $V^s(x;\phi) = V(x)$ but the solutions to (29) are a subset of those to (25). If (y^*, λ^*) is a saddle-point of (29), then there is $\phi^* \in \partial V(y^*)$ satisfying the first-order condition (26), that is,

$$\phi^* = -\beta^{-1} \left[D_y F(x, y^*) + \lambda^* D_y h(x, y^*) \right], \tag{30}$$

and the recursive equation $V^s(x;\phi) = F(x,y^*) + \lambda^* h(x,y^*) + \beta V^s(y^*;\phi^*)$ holds.

Using selective value function V^s we define *policy functions* $\varphi : X \times \Re^n \longrightarrow X \times \Re^n$ and $\ell : X \times \Re^n \longrightarrow \Re^k_+$ as time-invariant selections from solution to saddle-point Bellman equation (29) and the first-order condition (30). That is, $\varphi(x,\phi) = (y^*, \lambda^*)$ where (y^*, λ^*) is a saddle-point of (29), and $\ell(x,\phi) = \phi^*$ where $\phi^* \in \partial V(y^*)$ satisfies (30).

Let $\{x_t^*, \lambda_t^*, \phi_t^*\}_{t=1}^{\infty}$ be a sequence generated by policy functions (φ, ℓ) – that is, $(x_{t+1}^*, \lambda_{t+1}^*) = \varphi(x_t^*, \phi_t^*)$ and $\phi_{t+1}^* = \ell(x_t^*, \phi_t^*)$ for every $t \geq 0$, with initial state $x_0^* = x_0$ and co-state $\phi_0^* \in$

$\partial V(x_0)$. It follows from Proposition 4 that $\{x_t^*, \lambda_t^*\}$ is a saddle-point of (19). Sequence $\{x_t^*, \lambda_t^*, \phi_t^*\}$ can be found by solving a system of $n + n + k$ first-order difference equations. Those equations are (27), (28), and the complementary slackness conditions (21). The initial state and co-state are x_0 and an arbitrary $\phi_0 \in \partial V(x_0)$. Since the value function V is concave, it is almost everywhere differentiable implying that $\partial V(x_0) = DV(x_0)$ for almost every x_0.

Corollary 4 summarizes our results and reestablishes the link between the Bellman equation and the Euler equations, as in the differentiable case.

Corollary 4: *Suppose that conditions D1-D5 hold. If $\{x_t^*, \lambda_t^*, \phi_t^*\}_{t=1}^\infty$, with $\{x_t^*\} \in \ell_\infty^n$, is a sequence generated by policy functions (φ, ℓ), i.e. $(x_{t+1}^*, \lambda_{t+1}^*) = \varphi(x_t^*, \phi_t^*)$ and $\phi_{t+1}^* = \ell(x_t^*, \phi_t^*)$, with $x_0^* = x_0$ and $\phi_0^* \in \partial V(x_0)$, then $\{x_t^*, \lambda_t^*\}_{t=1}^\infty$ is a saddle-point of (19) at x_0. Furthermore, if at date $\tau + 1$ a new sequence $\{\widehat{x}_t^*, \widehat{\lambda}_t^*\}_{t=\tau+1}^\infty$ is generated using possibly different policy functions $(\widehat{\varphi}, \widehat{\ell})$ starting from initial state x_τ^* and co-state ϕ_τ^*, then $(\{x_t^*, \lambda_t^*\}_{t=1}^\tau, \{\widehat{x}_t^*, \widehat{\lambda}_t^*\}_{t=\tau+1}^\infty)$ is a saddle-point of (19) at x_0.*

Example 3 illustrates the results of this section.

Example 3 (A dynastic problem): consider a dynasty of overlapping generations who live for two periods. A household is formed by a young and an old member. In each period, the young decides for the household consumption for the next period. The allocation problem of the dynasty is to maximise the discounted utility of all future generations as follows:

$$\max_{\{x_t\}_{t=1}^\infty} \sum_{t=0}^\infty \beta^t \left[x_t + 2x_{t+1}\right] \tag{31}$$

$$\text{s.t. } x_t + x_{t+1} \le 4, \;\; x_{t+1} \le x_t, \;\; x_t \ge 0, \;\; t \ge 0,$$

for given $x_0 \in [0, 4]$. Let $V(x_0)$ be the value function of (31). It is easy to see that the value function is

$$V(x) = \begin{cases} 3x(1-\beta)^{-1} & \text{if } x \le 2 \\ 8 - x + 3(4-x)\beta(1-\beta)^{-1} & \text{if } x \ge 2. \end{cases} \tag{32}$$

The unique solution to (31) for $x_0 = 2$ is the constant sequence $x_t^* = 2$.

The value function V is concave. It is not differentiable at $x = 2$ where the super-differential is

$$\partial V(2) = [-(3\beta(1-\beta)^{-1} + 1), 3(1-\beta)^{-1}]. \tag{33}$$

The Euler equations for a saddle-point of (31) are

$$2-\lambda^*_{1,t}-\lambda^*_{2,t}+\beta\big[1-\lambda^*_{1,t+1}+\lambda^*_{2,t+1}\big]=0 \tag{34}$$

for $t\geq 1$, where $\lambda^*_{1,t}$ and $\lambda^*_{2,t}$ are the multipliers of the first and the second constraints in (31), respectively. For $x_0=2$ and the solution $x^*_t=2$, both constraints are binding. This implies that the slackness conditions are vacuous, and therefore the saddle-point multipliers are arbitrary positive solutions to difference equations (34) that satisfy the transversality condition (22).

The saddle-point Bellman equation is

$$V(x)=\text{SP}\min_{\lambda\geq 0}\max_{y}\left\{x+2y+\lambda_1(4-y-x)+\lambda_2(x-y)+\beta V(y)\right\}. \tag{35}$$

The unique (saddle-point) solution to (35) at $x=2$ is $y^*=2$. Starting from $x_0=2$, the saddle-point Bellman equation generates the sequence of solutions $x^*_t=2$. The first-order condition at x^*_t is

$$2-\lambda^*_{1,t+1}-\lambda^*_{2,t+1}+\beta\phi^*_{t+1}=0, \tag{36}$$

for some $\phi^*_{t+1}\in\partial V(x^*_{t+1})$. Equation (36) is the only restriction on multipliers generated by the saddle-point Bellman equation starting from $x_0=2$. The envelope theorem implies that $\phi^*_{t+1}=1-\hat{\lambda}_{1,t+2}+\hat{\lambda}_{2,t+2}$ for a date-$(t+2)$ multiplier $\hat{\lambda}_{t+2}$, but that multiplier can be different from λ^*_{t+2}. Given $\phi^*_{t+1}\in\partial V(2)$, the first-order condition (36) can be written more explicitly as

$$2-\beta\left(3\beta(1-\beta)^{-1}+1\right)\leq\lambda^*_{1,t+1}+\lambda^*_{2,t+1}\leq 2+3\beta(1-\beta)^{-1}.$$

This set is a superset of multipliers satisfying the Euler equation (34).

The envelope selection condition (27) is

$$\phi^*_t=1-\lambda^*_{1,t+1}+\lambda^*_{2,t+1}. \tag{37}$$

Clearly (37), combined with the first-order condition (36) implies the Euler equation (34). Thus the set of multipliers generated by the saddle-point Bellman equation and satisfying (37) is the same as the set of saddle-point multipliers of (31) in accordance with Propositions 3 and 4[11].

[11]Note that the envelope selection condition (37) in this example is a one-to-one correspondence between ϕ^*_t and λ^*_{t+1} and, therefore, the latter can be used as a co-state variable. This is so because functions F and h_i are additively separable.

In sum, the intertemporal Euler equation (20) is a *necessary* first-order condition for a solution to (19) - and therefore to (18) - and by Proposition 3 it is also *sufficient*. The same is true for a sequence of saddle-points of the Bellman equation (23) if the saddle-point multiplier is unique. If there are multiple multipliers, the system of Euler equations is a *sufficient but not necessary* condition for a sequence of saddle-points of the Bellman equation. However, for every sequence of solutions to a Bellman equation there exists a sequence of multipliers such that Euler equation is satisfied. Such a sequence can be generated recursively using policy functions from the selective value function and extending the state x_t^* with a co-state subgradient $\phi_t^* \in \partial V(x_t^*)$.

The problem of time-inconsistency of multipliers discussed in this section can result in non-optimal outcomes in the presence of forward-looking constraints. We address this problem now.

5 Recursive Contracts

In this section we extend the results of Section 4 to the recursive contracts of Marcet and Marimon (2017). Recursive contracts are dynamic optimization problems with *forward-looking* constraints. The presence of forward-looking constraints makes the standard methods of dynamic programming inapplicable. Marcet and Marimon show that recursive contracts can be solved by transforming optimization problems with forward-looking constraints into saddle-point problems. Solutions to a saddle-point Bellman equation are solutions to the original contracting problem provided that an *intertemporal consistency condition* is satisfied. The necessity of this consistency condition was motivated by an example given by Messner and Pavoni (2004), who showed that non-unique solutions to the saddle-point Bellman equation can fail to be solutions to the contracting problem. We show that imposing the envelope selection condition and extending the co-state generates time-consistent recursive saddle-point solutions. The envelope selection condition is shown to be equivalent to Marcet and Marimon's *intertemporal consistency condition*[12].

[12]Introduced in Marcet and Marimon (2015).

5.1 The partnership problem with limited enforcement

The deterministic partnership problem[13] with limited enforcement takes the form

$$\begin{aligned} V_\mu(y_0) &= \max_{\{c_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t \sum_{i=1}^{m} \mu_i \, u(c_{i,t}) && (38) \\ & \text{s.t.} \sum_{i=1}^{m} c_{i,t} \le \sum_{i=1}^{m} y_{i,t}, && (39) \\ & \sum_{n=0}^{\infty} \beta^n \, u(c_{i,t+n}) \ge v_i(y_{i,t}), && (40) \\ & c_{i,t} \ge 0, \ \text{ for all } i, t \ge 0, \end{aligned}$$

where the sequence $\{c_t\}$ is bounded, i.e. $\{c_t\} \in \ell_\infty^m$. The sequence of incomes $\{y_t\}$ follows a law of motion $y_{t+1} = g(y_t)$ for some $g : \Re_+^m \to \Re_+^m$ for given initial income vector y_0. We impose the following conditions:

P1. The function $u : \Re_+ \to \Re_+$ is increasing, concave, and differentiable and the functions and $v_i : \Re_+ \to \Re_+, i = 1, ..., m,$ are increasing, concave and continuous.

P2. Sequence $\{y_t\}$ is bounded, $\mu_i > 0$ for every i, and $\beta \in (0, 1)$.

P3. There exists $\{\hat{c}_t\} \in \ell_\infty^m$ with $\hat{c}_{i,t} > 0$ for every i and t such that constraints (39) and (40) hold with strict inequality.

A convenient way of analysing problem (38) is to consider the following constrained saddle-point problem resulting from adding forward-looking constraints (40) with multipliers to the objective function (see Marcet and Marimon (2017) for details):

$$\begin{aligned} \text{SP} \max_{\{c_t\}_{t=0}^\infty} \min_{\{\mu_t\}_{t=1}^\infty} \quad & \sum_{t=0}^{\infty} \beta^t \sum_{i=1}^{m} \left[\mu_{i,t+1} \left(u(c_{i,t}) - v_i(y_{i,t}) \right) + \mu_{i,t} v_i(y_{i,t}) \right] && (41) \\ & \text{s.t.} \sum_{i=1}^{m} c_{i,t} \le \sum_{i=1}^{m} y_{i,t} \\ & \mu_{i,t+1} \ge \mu_{i,t} \\ & c_{i,t} \ge 0, \ \text{ for all } i, t, \end{aligned}$$

[13] Marcet and Marimon develop their theory in a more general dynamic stochastic formulation. The approach presented in this section can be easily extended to their general setup.

where $\mu_0 = \mu$, and $\{c_t\} \in \ell_\infty^m$ and $\{\mu_t\} \in \ell_\infty^m$. The unconstrained saddle-point problem for the Lagrangian of (41) is

$$\text{SP} \max_{\{c_t,\lambda_{t+1}\}_{t=0}^\infty} \min_{\{\mu_t,\gamma_t\}_{t=1}^\infty} \sum_{t=0}^\infty \beta^t \sum_{i=1}^m \big\{\mu_{i,t+1}\left(u(c_{i,t}) - v_i(y_{i,t})\right) + \mu_{i,t} v_i(y_{i,t}) - \lambda_{i,t+1}(\mu_{i,t+1} - \mu_{i,t}) + \gamma_{t+1}\left(y_{i,t} - c_{i,t}\right)\big\}. \tag{42}$$

If $\{c_t^*\} \in \ell_\infty^m$ is a solution to (38), then there exists a bounded sequence[14] of weights $\{\mu_t^*\} \in \ell_\infty^m$ and summable sequences of multipliers $\{\lambda_t^*\} \in \ell_1^m$ and $\{\gamma_t^*\} \in \ell_1$ (see Dechert (1992)) such that $\{c_t^*, \lambda_t^*, \mu_t^*, \gamma_t^*\}$ is a saddle-point of (42). Conversely, if $\{c_t^*, \lambda_t^*, \mu_t^*, \gamma_t^*\}$ is a saddle-point of (42), then $\{c_t^*\}$ is a solution to (38).

The first-order necessary conditions with respect to $\mu_{i,t+1}$ for saddle-point $\{c_t^*, \lambda_t^*, \mu_t^*, \gamma_t^*\}$ of (42) are

$$u(c_{i,t}^*) - \left(v_i(y_{i,t}) + \lambda_{i,t+1}^*\right) + \beta\left(v_i(y_{i,t+1}) + \lambda_{i,t+2}^*\right) = 0 \tag{43}$$

for every i and $t \geq 0$. The respective condition with respect to $c_{i,t}$ is $\mu_{i,t+1}^* u'(c_{i,t}^*) = \gamma_{t+1}^*$. Equation (43) is the *intertemporal Euler equation* for the partnership problem. The Euler equation together with first-order conditions for $c_{i,t}$, the constraints, and complementary slackness conditions for γ_t^* and $\lambda_{i,t}^*$, are a system of first-order difference equations.

As in Section 4 (see Proposition 3), Euler equations and a transversality condition are sufficient conditions for a solution to (42).

Proposition 5: *Let* $\{c_t^*, \lambda_t^*, \mu_t^*, \gamma_t^*\}$*, with* $\{c_t^*\}_{t=0}^\infty \in \ell_\infty^m$, $\{\mu_t^*\}_{t=0}^\infty \in \ell_\infty^m$, $\mu_0^* = \mu$, $\lambda_t^* \geq 0$ *and* $\gamma_t^* \geq 0$ *for every* t*, satisfy the Euler equations (43), the first-order conditions w.r. to* $c_{i,t}$*, and the constraints and complementary slackness conditions (21). If the transversality condition*

$$\lim_{t\to\infty} \beta^t [v_i(y_{i,t}) + \lambda_{i,t+1}^*] = 0 \tag{44}$$

holds for every i*, then* $\{c_t^*, \lambda_t^*, \mu_t^*, \gamma_t^*\}$ *is a saddle-point of (42). In particular,* $\{c_t^*\}$ *is a solution to the partnership problem (38).*

Proof: see Appendix.

[14]In derivation of (41) weights μ_t^* obtain as partial sums of a summable sequence, and therefore are a convergent and hence bounded sequence.

Constrained saddle-point problem (41) has recursive structure that can be expressed by the following saddle-point Bellman equation:

$$W(\,y,\,\mu) = \text{SP} \min_{\tilde{\mu}} \max_{c,\lambda} \sum_{i=1}^{m} \left[\tilde{\mu}_i(u(c_i) - v_i(y_i)) + \mu_i v_i(y_i) - \lambda_i\left(\tilde{\mu}_i - \mu_i\right)\right] + \beta W(\tilde{y}, \tilde{\mu}) \tag{45}$$

$$\text{s.t.} \sum_{i=1}^{m} c_i \leq \sum_{i=1}^{m} y_i \tag{46}$$

$$c_i \geq 0,\ \lambda_i \geq 0,\ \text{ for all }\ i,$$

where $\tilde{y} = g(y)$. It follows from Theorem 3 in Marcet and Marimon (2017) that the value function V_μ satisfies equation (45), that is, $W(y_0, \mu) = V_\mu(y_0)$ for every $\mu \in \Re^m_+$. By a solution to the Bellman equation (45) we always mean a saddle-point $(c^*, \lambda^*, \tilde{\mu}^*, \gamma^*)$ where γ^* is a multiplier of constraint (46). The set of saddle-points of (45) is a product of two sets $M^*(y, \mu)$ and $N^*(y, \mu)$ so that $(c^*, \lambda^*) \in M^*(y, \mu)$ and $(\tilde{\mu}^*, \gamma^*) \in N^*(y, \mu)$ for every saddle-point $(c^*, \lambda^*, \tilde{\mu}^*, \gamma^*)$.

The value function W is homogeneous of degree one and convex in μ.[15] The envelope theorem for saddle-point problems, see (72) in the Appendix, when applied to the right-hand side of (45) implies that

$$\partial_\mu W(y,\,\mu) = v(y) + \{\lambda^* |\ (c^*, \lambda^*) \in M^*(y, \mu) \text{ for some } c^*\}. \tag{47}$$

Function W is differentiable with respect to μ at $(y,\,\mu)$ if and only if there is unique multiplier λ^* that is common to all solutions c^*. Since W is convex in μ, it is differentiable almost everywhere.

For every saddle-point $(c^*, \lambda^*, \tilde{\mu}^*, \gamma^*)$ of the Bellman equation (45) there exists subgradient vector $\phi^* \in \partial_\mu W(\tilde{y},\,\tilde{\mu}^*)$ such that the first-order conditions with respect to $\tilde{\mu}_i$

$$u(c_i^*) - (v_i(y_i) + \lambda_i^*) + \beta\phi_i^* = 0, \tag{48}$$

hold for every i.

For every sequence $\{c_t^*, \lambda_t^*, \mu_t^*, \gamma_t^*\}$ which is a saddle-point of (42), $(c_t^*, \lambda_{t+1}^*, \mu_{t+1}^*, \gamma_{t+1}^*)$ is a saddle-point of the Bellman equation at (y_t, μ_t^*) for every t. It follows that for every solution $\{c_t^*\}$ to partnership problem (38), there exist weights μ_{t+1}^* such that (c_t^*, μ_{t+1}^*) is a solution to the Bellman equation for every t. As in Proposition 4, the converse result holds if the envelope selection condition is satisfied.

[15] See Marcet and Marimon (2017) for discussion of the homogeneity properties of W with respect to μ.

Proposition 6: *Suppose that conditions P1-P3 hold. Let* $\{c_t^*, \lambda_t^*, \mu_t^*, \gamma_t^*\}$ *be a sequence of saddle-points generated by saddle-point Bellman equation (45) starting at* (y_0, μ), *with* $\{c_t^*\}_{t=0}^\infty \in \ell_\infty^m$ *and* $\{\mu_t^*\}_{t=0}^\infty \in \ell_\infty^m$, *and let* $\{\phi_t^*\}_{t=0}^\infty$ *be the corresponding sequence of subgradients satisfying (48), with* $\phi_t^* \in \partial_\mu W(y_t, \mu_t^*)$ *for all* t. *If the following envelope selection condition*

$$\phi_{i,t}^* = v_i(y_{i,t}) + \lambda_{i,t+1}^* \tag{49}$$

holds for every i *and every* $t \geq 0$, *then* $\{c_t^*, \lambda_t^*, \mu_t^*, \gamma_t^*\}$ *is a saddle-point of (42)*[16].

Proof: The first-order conditions for saddle-point $(c_t^*, \lambda_{t+1}^*, \mu_{t+1}^*, \gamma_{t+1}^*)$ of Bellman equation (45) at (y_t, μ_t^*) are

$$u(c_{i,t}^*) - \left(v_i(y_{i,t}) + \lambda_{i,t+1}^*\right) + \beta\phi_{i,t+1}^* = 0 \tag{50}$$

and $\mu_{i,t+1}^* u'(c_{i,t}^*) = \gamma_{t+1}^*$ for every i. Eq. (50) together with the envelope selection condition (49) imply the Euler equation (43). The complementary slackness conditions follow from respective conditions for (45). Since W is convex in μ, and $\{y_t\}$ and $\{\mu_t^*\}$ are bounded, the sequence $\{\phi_t^*\}$ is bounded and hence the transversality condition $\lim_{t\to\infty} \beta^t \phi_t^* = 0$ holds. The conclusion follows now from Proposition 5. □

Proposition 6 corresponds to Theorem 4 in Marcet and Marimon (2017) where the envelope selection condition (49) is replaced by the *intertemporal consistency condition*

$$\phi_{i,t}^* = u(c_{i,t}^*) + \beta\phi_{i,t+1}^*. \tag{51}$$

Because of the first-order condition (50), those two conditions are equivalent. Note that condition (51) and the transversality condition $\lim_{t\to\infty} \beta^t \phi_{i,t}^* = 0$ imply that

$$\phi_{i,t}^* = \sum_{n=0}^{\infty} \beta^n u(c_{i,t+n}^*).$$

Using (49) and $\lambda_{i,t+1}^* \geq 0$, it follows that

$$\sum_{n=0}^{\infty} \beta^n u(c_{i,t+n}^*) \geq v_i(y_{i,t}). \tag{52}$$

[16]Note that recursive contracts are additively separable in μ_t^* and μ_{t+1}^* and, therefore, λ_{t+1}^* can be used as a co-state variable (see footnote 11).

That is, the intertemporal consistency condition – equivalently, the envelope selection condition – implies that the participation constraints (40) hold.

As in Section 4 (see Corollary 4), the *envelope selection condition* (49) guarantees *consistency* of multipliers and solutions generated by the saddle-point Bellman equation. It can be dispensed with if the saddle-point multiplier is unique, which is a necessary and sufficient condition for value function W to be differentiable in μ. Inconsistency of multipliers may occur if, given $(c^*_{t-1}, \lambda^*_t, \mu^*_t, \gamma^*_t)$ with the corresponding subgradient $\phi^*_t \in \partial_\mu W(y_t, \mu^*_t)$ from the first-order condition (50) at $t-1$, multiplier λ^*_{t+1} is chosen without satisfying envelope selection condition (49) at t, which is likely to happen if the saddle-point Bellman equation is solved only knowing (y_t, μ^*_t). Then the Euler equation (43) is not satisfied for λ^*_t and λ^*_{t+1}, and the sequence $\{c^*_t, \lambda^*_t, \mu^*_t, \gamma^*_t\}$ need not be a saddle-point of (42). Because the multiplier λ^*_{t+1} has to be chosen together with consumption c^*_t in the set $M^*(y_t, \mu^*_t)$, inconsistent choice of the multiplier may lead to consumption that is either suboptimal, or violates the participation constraints.

We define the *selective value function* of state y and co-state (μ, ϕ) for $\phi \in \partial_\mu W(y, \mu)$ in a similar way as in Section 4, that is:

$$W^s(\, y,\ \mu; \phi) = \text{SP} \min_{\tilde{\mu}} \max_{c,\lambda} \sum_{i=1}^{m} \left[\tilde{\mu}_i(u(c_i) - v_i(y_i)) + \mu_i v_i(y_i) - \lambda_i \left(\tilde{\mu}_i - \mu_i\right)\right] + \beta W(\tilde{y}, \tilde{\mu}) \tag{53}$$

$$\text{s.t.}\ \ \phi_i = v(y_i) + \lambda_i, \tag{54}$$

$$\sum_{i=1}^{m} c_i \leq \sum_{i=1}^{m} y_i,$$

$$c_i \geq 0, \lambda_i \geq 0, \ \text{ for all } \ i,$$

where $\tilde{y} = g(y)$.[17] It holds that $W^s(\, y,\ \mu; \phi) = W(\, y,\ \mu)$, but the (saddle-point) solutions in (53) and in (45) can be different. The first-order condition for saddle-point $(c^*, \lambda^*, \tilde{\mu}^*, \gamma^*)$ of (53) with respect to μ is

$$u(c^*_i) - (v_i(y_i) + \lambda^*_i) + \beta \phi^*_i = 0, \tag{55}$$

where $\phi^* \in \partial_\mu W(\tilde{y}, \mu^*)$. It holds that

$$W^s(\, y,\ \mu; \phi) = \sum_{i=1}^{m} \left[\tilde{\mu}^*_i(u(c^*_i) - v_i(y_i)) + \mu_i v_i(y_i)\right] + \beta W^s(\tilde{y}, \tilde{\mu}^*; \phi^*).$$

[17]Note that the multiplier λ is uniquely determined by constraint (54).

The policy functions $\varphi : \Re_+^{3m} \to \Re_+^{2m}$ and $\ell : \Re_+^{3m} \to \Re_+^{m+1}$ are defined by $\varphi(y, \mu, \phi) = (c^*, \tilde{\mu}^*, \lambda^*, \gamma^*)$ such that $(c^*, \tilde{\mu}^*, \lambda^*, \gamma^*)$ is a saddle-point of (53) and $\ell(y, \mu, \phi) = \phi^*$ where $\phi^* \in \partial_\mu W(\tilde{y}, \mu^*)$ satisfies (55).

Policy functions (φ, ℓ) can be used to generate sequences of saddle-points $\{c_t^*, \lambda_t^*, \mu_t^*, \gamma_t^*\}$ and subgradients $\{\phi_t^*\}$ such that $(c_t^*, \lambda_{t+1}^*, \mu_{t+1}^*, \gamma_{t+1}^*) = \varphi(y_t, \mu_t^*, \phi_t^*)$ and $\phi_{t+1}^* = \ell(y_t, \mu_t^*, \phi_t^*)$ for every $t \geq 0$, with initial state y_0 and co-state (μ_0^*, ϕ_0^*) where $\mu_0^* = \mu$ and $\phi_0^* \in \partial_\mu W(y_0, \mu)$. It follows from Proposition 5 that $\{c_t^*, \lambda_t^*, \mu_t^*, \gamma_t^*\}_{t=0}^\infty$ is a saddle-point of (42). Sequence $\{c_t^*, \lambda_t^*, \mu_t^*, \gamma_t^*, \phi_t^*\}$ can be found by solving system equations (49) and (50) together with first-order conditions w.r. $c_{i,t}$ and complementary slackness conditions. All these equations are first-order difference equations.

Corollary 5 which follows summarizes our results for recursive contracts. It generalizes the main sufficiency result of Marcet and Marimon (2017; Theorem 4 and its Corollary) to the case where solutions may not be unique (the value function W may not be differentiable).

Corollary 5: *Suppose that conditions P1-P3 hold. If $\{c_t^*, \lambda_t^*, \mu_t^*, \gamma_t^*\}$ and $\{\phi_t^*\}$ are sequences of saddle-points and subgradients generated by policy functions (φ, ℓ) starting from $\mu_0^* = \mu$ and $\phi_0^* \in \partial_\mu W(y_0, \mu)$, with $\{c_t^*\}_{t=0}^\infty \in \ell_\infty^m$ and $\{\mu_t^*\}_{t=0}^\infty \in \ell_\infty^m$, then $\{c_t^*, \lambda_t^*, \mu_t^*, \gamma_t^*\}$ is a saddle-point of (42) at μ. Furthermore, if at date $\tau+1$ a new sequence $\{\widehat{c}_t^*, \widehat{\lambda}_t^*, \widehat{\mu}_t^*, \widehat{\gamma}_t^*\}_{t=\tau+1}^\infty$ is generated using possibly different policy functions $(\widehat{\varphi}, \widehat{\ell})$ starting from initial state y_τ and co-state $(\mu_\tau^*, \phi_\tau^*)$, then $(\{c_t^*, \lambda_t^*, \mu_t^*, \gamma_t^*\}_{t=1}^\tau, \{\widehat{c}_t^*, \widehat{\lambda}_t^*, \widehat{\mu}_t^*, \widehat{\gamma}_t^*\}_{t=\tau+1}^\infty)$ is also a saddle-point of (42) at μ.*

Because the objective and the constraint functions are additively separable in the partnership problem, the envelope selection condition (49) is a one-to-one relation between subgradient ϕ_t^* and multiplier λ_{t+1}^* and hence the multiplier can be used as co-state in the recursive method of Corollary 5. More precisely, the co-state can be $(\mu_t^*, \lambda_{t+1}^*)$ instead of (μ_t^*, ϕ_t^*).

In sum, when using an infinite-horizon approach, the intertemporal Euler equation (43) along with other first-order conditions are necessary and sufficient for a solution to the partnership problem (38). Using a dynamic programming approach, the envelope selection condition is a necessary and sufficient condition for a sequence of saddle-points of the Bellman equation to be a solution to the partnership problem. Such sequence can be generated recursively using a policy function from the selective value function. As in Section 4, we provide an example that illustrates the results of this section.

Example 4 (Messner and Pavoni (2004)): consider a partnership problem (38) with two agents and linear utilities:

$$\begin{aligned} W(\mu) &= \max_{\{c_t\}_0^\infty} \sum_{t=0}^{\infty} \beta^t \sum_{i=1}^{2} \mu_i c_{i,t} && (56) \\ & \text{s.t. } c_{1,t} + c_{2,t} \leq y, \\ & \quad \sum_{j=0}^{\infty} \beta^j c_{1,t+j} \geq 0, \ \sum_{j=0}^{\infty} \beta^j c_{2,t+j} \geq b(1-\beta)^{-1}, && (57) \\ & \quad c_{i,t} \geq 0, \ i = 1,2, \ \text{for all } t \geq 0, \end{aligned}$$

where $0 < b < y$, $\mu_i > 0$ for $i = 1, 2$, and $0 < \beta < 1$. Note that agent 1's participation constraint is not binding.

The value function is

$$W(\mu) = \begin{cases} (1-\beta)^{-1}[\mu_1(y-b) + \mu_2 b] & \text{if } \mu_1 \geq \mu_2 \\ (1-\beta)^{-1}\mu_2 y & \text{if } \mu_1 \leq \mu_2. \end{cases} \tag{58}$$

If $\mu_1 > \mu_2$, the constant sequence $c^*_{1,t} = y - b$ and $c^*_{2,t} = b$ is a solution to (56), but we shall see that there are many other solutions. The value function W is convex and differentiable if $\mu_1 \neq \mu_2$, but it is not differentiable if $\mu_1 = \mu_2$, where the sub-differential is

$$\partial W(\mu) = (1-\beta)^{-1} co\{(y-b, b), (0, y)\}. \tag{59}$$

The saddle-point Bellman equation (45) is

$$W(\mu) = \text{SP } \min_{\tilde{\mu}} \max_{c,\lambda} \left\{ \sum_{i=1}^{2} \left[\tilde{\mu}_i c_i - \lambda_i \left(\tilde{\mu}_i - \mu_i\right)\right] - (\tilde{\mu}_2 - \mu_2) b (1-\beta)^{-1} + \beta W(\tilde{\mu}) \right\} \tag{60}$$

$$\begin{aligned} & \text{s.t. } c_1 + c_2 \leq y, \\ & \quad c_i \geq 0, \ \lambda_i \geq 0, \ i = 1,2. \end{aligned}$$

A sequence of saddle-points $\{\mu_t^*, c_t^*, \lambda_t^*, \gamma_t^*\}$ of (60) generated by a policy function can be found by recursively solving a system of equations consisting of the first-order conditions

$$\mu_{i,t+1}^* - \gamma_{t+1}^* = 0, \tag{61}$$

$$c_{1,t}^* - \lambda_{1,t+1}^* + \beta\phi_{1,t+1}^* = 0 \tag{62}$$

$$c_{2,t}^* - (b(1-\beta)^{-1} + \lambda_{2,t+1}^*) + \beta\phi_{2,t+1}^* = 0, \tag{63}$$

with $\phi_{t+1}^* \in \partial W(\mu_{t+1}^*)$, the complementary slackness conditions for λ_{t+1}^*, and the envelope selection conditions

$$\phi_{1,t}^* = \lambda_{1,t+1}^* \tag{64}$$

$$\phi_{2,t}^* = b(1-\beta)^{-1} + \lambda_{2,t+1}^*. \tag{65}$$

Suppose that the initial state is $\mu_0^* = \mu$ such that $\mu_1 > \mu_2$. Since W is differentiable at μ_0^*, the initial co-state is $\phi_0^* = DW(\mu_0^*)$, that is, $\phi_{1,0}^* = (y-b)(1-\beta)^{-1}$ and $\phi_{2,0}^* = b(1-\beta)^{-1}$. From equations (64 - 65) we obtain $\lambda_{1,1}^* = (y-b)(1-\beta)^{-1}$ and $\lambda_{2,1}^* = 0$, and using complementary slackness $\mu_{1,1}^* = \mu_1$ and $\mu_{2,1}^* = \mu_{1,1}^*$. Because W is not differentiable at μ_1^*, $\phi_{1,1}^*$ and $\phi_{2,1}^*$ can be arbitrary satisfying $\phi_{1,1}^* + \phi_{2,1}^* = y(1-\beta)^{-1}$, $\phi_{1,1}^* \geq 0$ and $\phi_{2,1}^* \geq b(1-\beta)^{-1}$ (see (59)) and consumption plan c_0^* can be any selection satisfying $c_{1,0}^* = (y-b)(1-\beta)^{-1} - \beta\phi_{1,1}^*$ and $c_{2,0}^* = b(1-\beta)^{-1} - \beta\phi_{2,1}^*$. Selection of c_0^* determines ϕ_1^*. We have thus derived $\phi_1^* = \ell(\mu_0^*, \phi_0^*)$, and $(c_0^*, \mu_1^*, \lambda_1^*) = \varphi(\mu_0^*, \phi_0^*)$ for a policy function (φ, ℓ).[18]

Next, iteration of equations (61 - 65) with given state μ_1^* and co-state ϕ_1^* gives $\mu_2^* = \mu_1^*,\ c_{1,1}^* = \phi_{1,1}^* - \beta\phi_{1,2}^*,\ c_{2,1}^* = \phi_{2,1}^* - \beta\phi_{2,2}^*$ where $\phi_{1,2}^* + \phi_{2,2}^* = y(1-\beta)^{-1}$, $\phi_{1,2}^* \geq 0$ and $\phi_{2,2}^* \geq b(1-\beta)^{-1}$. Again, a selection of c_1^* determines ϕ_2^*. With this step, we have derived $\phi_2^* = \ell(\mu_1^*, \phi_1^*)$, and $(c_1^*, \mu_2^*, \lambda_2^*) = \varphi(\mu_1^*, \phi_1^*)$. All subsequent iterations follow the same pattern with $\mu_t^* = \mu_1^*$ for all $t > 1$, and μ_t^* being the point of non-differentiability of value function W. Time-invariant policy function selects a stationary solution to the equations for $t \geq 2$. Corollary 5 implies that every sequence generated in this way is an optimal solution to (56). For example, consumption sequence $c_{1,t}^* = y - b$ and $c_{2,t}^* = b$ is optimal and can be generated by a policy function with $\lambda_{1,t}^* = (y-b)(1-\beta)^{-1}$, $\lambda_{2,t}^* = 0$, $\phi_{1,t}^* = (y-b)(1-\beta)^{-1}$ and $\phi_{2,t}^* = b(1-\beta)^{-1}$ for $t \geq 1$. One can easily verify that a sequence $c_{1,0}^* = y-b+\Delta,\ c_{2,0}^* = b-\Delta$, $c_{1,1}^* = y-b-\frac{1}{\beta}\Delta,\ c_{2,1}^* = b+\frac{1}{\beta}\Delta$, and $c_{1,t}^* = y-b$ and $c_{2,t}^* = b$ for $t \geq 2$ is optimal, too, for any small positive Δ.

[18] To simplify notation, we eliminated multipliers γ_t^* (given by (61)) from consideration.

We show next that a sequence of saddle-points of the Bellman equation (60) may not be an optimal solution to (56) if the envelope selection conditions (64, 65) are not satisfied. Consider the same sequence of weights $\{\mu_t^*\}$ as before and a sequence of consumptions, multipliers, and subgradients given by $c_{1,t}^* = 0,\ c_{2,t}^* = y$ for $t \geq 0$, $\lambda_{1,t}^* = 0$, $\lambda_{2,t}^* = (y-b)(1-\beta)^{-1}$ for $t \geq 1,\ \phi_{1,t}^* = 0$ and $\phi_{2,t}^* = y(1-\beta)^{-1}$ for $t \geq 1$, and $\phi_0^* = DW(\mu_0^*)$. Note that $\phi_t^* \in \partial W(\mu_t^*)$ for all t. This sequence satisfies first-order conditions (61 - 63). However, envelope selection conditions (64, 65) are violated for $t = 1$ implying that the Euler equation is violated at date 0. This consumption sequence is not optimal since there is excessive consumption for agent 2.

Similarly, if $\beta y \geq b$, then the sequence given by $c_{1,t}^* = y$, $c_{2,t}^* = 0$ for $t \geq 0$, $\lambda_{1,t}^* = (y-b)(1-\beta)^{-1}$, $\lambda_{2,t}^* = 0$ for $t \geq 1,\ \phi_{1,t}^* = (\beta y - b)[\beta(1-\beta)]^{-1},\ \phi_{2,t}^* = b[\beta(1-\beta)]^{-1}$ for $t \geq 1$, and $\phi_0^* = DW(\mu_0^*)$ satisfies equations (61 - 62) and also $\phi_t^* \in \partial W(\mu_t^*)$. Here, envelope selection conditions (64, 65) are violated at every $t \geq 1$. This consumption sequence does not satisfy participation constraints (57).

6 Conclusions

This paper makes three contributions to constrained optimisation problems when the value function may be non-differentiable due to the presence of binding constraints. First, it extends the envelope theorem by providing a novel characterization of the super- and sub-differentials for concave and convex value functions. Second, it uncovers a previously unknown form of time-inconsistency in standard dynamic constrained optimisation problems: restarting the Bellman equation at a later state, say x_t^*, may result in time-inconsistent Lagrange multipliers for which the Euler equations fail to hold and the continuation saddle-point is not part of a saddle-point of the optimisation problem from x_0. Nevertheless, the solutions to the Bellman equation remain time-consistent in the standard dynamic optimization problems. In the presence of *forward-looking constraints*, the time-inconsistency of multipliers of the Bellman equation can turn into time-inconsistent solutions. Our third contribution is a method of restoring time-consistency of multipliers and/or solutions by imposing an *envelope consistency condition*. The method extends the co-state and introduces a *selective value function* to recover the link between the solutions to the Bellman and Euler equations. The method is superior to the existing computational techniques of solving dynamic models with binding constraints that often adopt relatively ad-hoc procedures to account for non-linearities and possible time-inconsistencies caused by the presence of such constraints[19].

Appendix

A. Saddle-point problems

We extend results of Sections 2 and 3.1 to saddle-point problems.

Consider the following parametric saddle-point problem

$$\text{SP} \max_{y\in Y} \min_{z\in Z} f(x,y,z) \tag{66}$$

subject to

$$h_i(x,y) \geq 0, \quad g_i(x,z) \leq 0, \quad i = 1,\ldots,k \tag{67}$$

with parameter $x \in \Re^m$. Let $V(x)$ denote the value function of the problem (66–67). The Lagrangian function associated with (66–67) is

$$\mathcal{L}(x,y,z,\lambda,\gamma) = f(x,y,z) + \lambda h(x,y) + \gamma g(x,z), \tag{68}$$

where $\lambda \in \Re^k_+$ and $\gamma \in \Re^k_+$ are vectors of multipliers. A saddle-point of $\mathcal{L}$ is vector $(y^*, z^*, \lambda^*, \gamma^*)$ where $\mathcal{L}$ is maximized with respect to $y \in Y$ and $\gamma \in \Re^k_+$, and minimized with respect to $z \in Z$ and $\lambda \in \Re^k_+$. The set of saddle-points of $\mathcal{L}$ is a product of two sets M^* and N^* so that $(y^*, \gamma^*) \in M^*$ and $(z^*, \lambda^*) \in N^*$ for every saddle-point $(y^*, z^*, \lambda^*, \gamma^*)$. If $(y^*, z^*, \lambda^*, \gamma^*)$ is a saddle-point, then (y^*, z^*) is a solution to (66–67).

For single-dimensional parameter x, the directional derivatives of the value function V at $x \in intX$ are

$$V'(x+) = \max_{(y^*,\gamma^*)\in M^*} \min_{(z^*,\lambda^*)\in N^*} \Big[\frac{\partial f}{\partial x}(x,y^*,z^*) + \lambda^* \frac{\partial h}{\partial x}(x,y^*) + \gamma^* \frac{\partial g}{\partial x}(x,z^*)\Big] \tag{69}$$

and

$$V'(x-) = \min_{(y^*,\gamma^*)\in M^*} \max_{(z^*,\lambda^*)\in N^*} \Big[\frac{\partial f}{\partial x}(x,y^*,z^*) + \lambda^* \frac{\partial h}{\partial x}(x,y^*) + \gamma^* \frac{\partial g}{\partial x}(x,z^*)\Big] \tag{70}$$

[19] For example, Guerrieri and Iacovello (2015) consider separate regimes where constraints are binding or not binding.

where the order of maximum and minimum does not matter. Corollary 1 can be easily extended to this case.

Suppose that multi-dimensional parameter x can be decomposed in $x = (x^1, x^2)$ so that the constraints in the saddle-point problem (66–67) can be written as

$$h_i(x^1, y) \geq 0, \quad g_i(x^2, z) \leq 0, \quad i = 1, \dots, k. \tag{71}$$

If function f is concave in x^1 and convex in x^2, while the functions h_i are concave in $x^1$and y and the functions g_i are convex in x^2 and z, then $V(x^1, x^2)$ is concave in x^1 and convex in x^2. The super-sub-differential calculus of Section 3.2 can be extended to this class of saddle-point problems. For instance, the subdifferential of value function V with respect to x^2 is

$$\partial V_{x^2}(x) = \bigcap_{(z^*,\lambda^*)\in N^*} co \left\{ \bigcup_{(y^*,\gamma^*)\in M^*} \{ D_{x^2} f(x, y^*, z^*) + \lambda^* D_{x^2} h(x, y^*) + \gamma^* D_{x^2} g(x, z^*) \} \right\}. \tag{72}$$

B. Proofs

We first prove the following Lemma:

Lemma 1. Under A1-A4, the sets $Y^*(x)$ and $\Lambda^*(x)$ are compact for every $x \in X$. Further, the correspondences Y^* and Λ^* are upper hemi-continuous on X.

Proof:

Assumptions A1 and A2 imply that the value $V(x)$ of the optimization problem (1–2) is well defined. If λ^* is a saddle-point multiplier at x, then, by saddle-point property (4), it holds

$$f(x, \hat{y}_i) + \lambda^* h(x, \hat{y}_i) \leq V(x). \tag{73}$$

Using A3, it follows from (73) that

$$\lambda_i^* \leq \frac{V(x) - f(x, \hat{y}_i)}{h_i(x, \hat{y}_i)}. \tag{74}$$

Using $\bar{\lambda}_i(x)$ to denote the RHS of (74), we conclude that the domain $\Re^k_+$ of multipliers in saddle-point problem (4) can be replaced by the compact set $\times_{i=1}^k [0, \bar{\lambda}_i(x)]$. This implies that the set of saddle-point multipliers $\Lambda^*(x)$ is compact. The set of saddle-point solutions $Y^*(x)$ is compact, too. The Maximum Theorem implies that correspondences Y^* and Λ^* are upper hemi-continuous on X. □

Proof of Theorem 1:

We shall prove that equations (8) and (9) hold for arbitrary $x_0 \in intX$. Let $\Delta f(t, y)$ denote the difference quotient of function f with respect to x at x_0, that is

$$\Delta f(t, y) = \frac{f(x_0 + t, y) - f(x_0, y)}{t}$$

for $t \neq 0$. For $t = 0$, we set $\Delta f(0, y) = \frac{\partial f}{\partial x}(x_0, y)$. Assumptions of Theorem 1 imply that function $\Delta f(t, y)$ is continuous in (t, y) on $Y \times \{X - x_0\}$.

Similar notation $\Delta h_i(t, y)$ is used for each function h_i, and $\Delta\mathcal{L}(t, y, \lambda)$ for the Lagrangian. Functions $\Delta h_i(t, y)$ are continuous in (t, y). Note that $\Delta\mathcal{L}(t, y, \lambda) = \Delta f(t, y) + \lambda\Delta h(t, y)$, where we use the scalar-product notation $\lambda\Delta h(t, y) = \sum_i \lambda_i \Delta h_i(t, y)$.

Saddle-point property (4) together with (5) imply that

$$V(x_0 + t) \geq \mathcal{L}(x_0 + t, y_0^*, \lambda_t^*) \tag{75}$$

and

$$V(x_0) \leq \mathcal{L}(x_0, y_0^*, \lambda_t^*) \tag{76}$$

for every $\lambda_t^* \in \Lambda^*(x_0 + t)$ and $y_t^* \in Y^*(x_0 + t)$. Subtracting (76) from (75) and dividing the result on both sides by $t > 0$, we obtain

$$\frac{V(x_0 + t) - V(x_0)}{t} \geq \Delta\mathcal{L}(t, y_0^*, \lambda_t^*) = \Delta f(t, y_0^*) + \lambda_t^* \Delta h(t, y_0^*). \tag{77}$$

Since (77) holds for every $y_0^* \in Y^*(x_0)$, we can take the maximum on the right-hand side and obtain

$$\frac{V(x_0 + t) - V(x_0)}{t} \geq \max_{y_0^* \in Y^*(x_0)} [\Delta f(t, y_0^*) + \lambda_t^* \Delta h(t, y_0^*)]. \tag{78}$$

Consider function Ψ defined as

$$\Psi(t, \lambda) = \max_{y_0^* \in Y^*(x_0)} [\Delta f(t, y_0^*) + \lambda\Delta h(t, y_0^*)] \tag{79}$$

so that the expression on the right-hand side of (78) is $\Psi(t, \lambda_t^*)$. Since $Y^*(x_0)$ is compact by Lemma 1, it follows from the Maximum Theorem that Ψ is a continuous function of (t, λ).

Further, since $\lambda_t^* \in \Lambda^*(x_0+t)$ and Λ^* is an upper hemi-continuous correspondence by Lemma 1, we obtain

$$\liminf_{t\to 0+} \Psi(t,\lambda_t^*) \geq \min_{\lambda_0^*\in\Lambda^*(x_0)} \Psi(0,\lambda_0^*) = \min_{\lambda_0^*\in\Lambda^*(x_0)} \max_{y_0^*\in Y^*(x_0)} [\frac{\partial f}{\partial x}(x_0,y_0^*) + \lambda_0^* \frac{\partial h}{\partial x}(x_0,y_0^*)] \quad (80)$$

where we used the scalar-product notation $\lambda_0^* \frac{\partial h}{\partial x} = \sum_i \lambda_{i0}^* \frac{\partial h_i}{\partial x}$. It follows from (80) and (78) that

$$\liminf_{t\to 0+} \frac{V(x_0+t)-V(x_0)}{t} \geq \min_{\lambda_0^*\in\Lambda^*(x_0)} \max_{y_0^*\in Y^*(x_0)} \big[\frac{\partial f}{\partial x}(x_0,y_0^*) + \lambda_0^* \frac{\partial h}{\partial x}(x_0,y_0^*)\big]. \quad (81)$$

We also have

$$V(x_0+t) \leq \mathcal{L}(x_0+t, y_t^*, \lambda_0^*) \quad (82)$$

and

$$V(x_0) \geq \mathcal{L}(x_0, y_t^*, \lambda_0^*) \quad (83)$$

which together imply

$$\frac{V(x_0+t)-V(x_0)}{t} \leq \Delta f(t,y_t^*) + \lambda_0^* \Delta h(t,y_t^*) \quad (84)$$

for $t>0$. Taking the minimum over $\lambda_0^* \in \Lambda^*(x_0)$ on the right-hand side of (84) results in

$$\frac{V(x_0+t)-V(x_0)}{t} \leq \min_{\lambda_0^*\in\Lambda^*(x_0)} [\Delta f(t,y_t^*) + \lambda_0^* \Delta h(t,y_t^*)]. \quad (85)$$

Consider function Φ defined as

$$\Phi(t,y) = \min_{\lambda_0^*\in\Lambda^*(x_0)} [\Delta f(t,y) + \lambda_0^* \Delta h(t,y)]$$

so that the expression on the right-hand side of (85) is $\Phi(t,y_t^*)$. It follows from the Maximum Theorem that Φ is a continuous function of (t,y). Using upper hemi-continuity of correspondence Y^* (see Lemma 1), we obtain

$$\limsup_{t\to 0+} \Phi(t,y_t^*) \leq \max_{y_0^*\in Y^*(x_0)} \Phi(0,y_0^*) = \max_{y_0^*\in Y^*(x_0)} \min_{\lambda_0^*\in\Lambda^*(x_0)} [\frac{\partial f}{\partial x}(x_0,y_0^*) + \lambda_0^* \frac{\partial h}{\partial x}(x_0,y_0^*)]. \quad (86)$$

It follows now from (86) and (85) that

$$\limsup_{t\to 0+} \frac{V(x_0+t)-V(x_0)}{t} \leq \max_{y_0^*\in Y^*(x_0)} \min_{\lambda_0^*\in\Lambda^*(x_0)} \big[\frac{\partial f}{\partial x}(x_0,y_0^*) + \lambda_0^* \frac{\partial h}{\partial x}(x_0,y_0^*)\big]. \quad (87)$$

It holds (see Lemma 36.1 in Rockafellar (1970)) that

$$\max_{y_0^* \in Y^*(x_0)} \min_{\lambda_0^* \in \Lambda^*(x_0)} \big[\frac{\partial f}{\partial x}(x_0, y_0^*) + \lambda_0^* \frac{\partial h}{\partial x}(x_0, y_0^*)\big] \leq \min_{\lambda_0^* \in \Lambda^*(x_0)} \max_{y_0^* \in Y^*(x_0)} \big[\frac{\partial f}{\partial x}(x_0, y_0^*) + \lambda_0^* \frac{\partial h}{\partial x}(x_0, y_0^*)\big]. \tag{88}$$

It follows from (81), (87) and (88) that the right-hand side derivative $V'(x_0+)$ exists and is given by

$$V'(x_0+) = \max_{y_0^* \in Y^*(x_0)} \min_{\lambda_0^* \in \Lambda^*(x_0)} \big[\frac{\partial f}{\partial x}(x_0, y_0^*) + \lambda_0^* \frac{\partial h}{\partial x}(x_0, y_0^*)\big]$$

where the order of maximum and minimum does not matter. This establishes eq. (8) of Theorem 1. The proof of (9) is similar. □

Proof of Theorem 2: By Theorem 23.2 in Rockafellar (1970), $\phi \in \partial V(x_0)$ if and only if $V'(x_0; \hat{x})\phi \leq \hat{x}\phi$ for every $\hat{x}$ such that $x_0 + \hat{x} \in X$. Applying (10), we obtain that $\phi \in \partial V(x_0)$ if and only if

$$\min_{\lambda_0^* \in \Lambda^*(x_0)} \big[D_x f(x_0, y_0^*) + \sum_{i=1}^{k} \lambda_{i0}^* D_x h_i(x_0, y_0^*)\big]\, \hat{x} \leq \phi \hat{x} \quad \text{for every } \hat{x}, \tag{89}$$

for every $y_0^* \in Y^*(x_0)$, where we used the fact that inequality (89) holds for every y_0^* if and only if it holds for the maximum over y_0^*. The left-hand side of (89), as a function of $\hat{x}$, is the negative of the support function of the set

$$\bigcup_{\lambda_0^* \in \Lambda^*(x_0)} \{D_x f(x_0, y_0^*) + \sum_{i=1}^{k} \lambda_{i0}^* D_x h_i(x_0, y_0^*)\}. \tag{90}$$

Since $\Lambda^*(x_0)$ is convex and compact, the set (90) is compact and convex. Theorem 13.1 in Rockafellar (1970) implies that (89) is equivalent to

$$\phi \in \bigcup_{\lambda_0^* \in \Lambda^*(x_0)} \{D_x f(x_0, y_0^*) + \sum_{i=1}^{k} \lambda_{i0}^* D_x h_i(x_0, y_0^*)\}$$

for every $y_0^* \in Y^*(x_0)$. Consequently, $\phi \in \partial V(x_0)$ if and only if

$$\phi \in \bigcap_{y_0^* \in Y^*(x_0)} \bigcup_{\lambda_0^* \in \Lambda^*(x_0)} \{D_x f(x_0, y_0^*) + \sum_{i=1}^{k} \lambda_{i0}^* D_x h_i(x_0, y_0^*)\}.$$

□

Proof of Corollary 2: If $\Lambda^*(x_0)$ is a singleton set or h_i does not depend on x for every i, then

$$\bigcup_{\lambda_0^* \in \Lambda^*(x_0)} \{D_x f(x_0, y_0^*) + \sum_{i=1}^{k} \lambda_{i0}^* D_x h_i(x_0, y_0^*)\}$$

is a singleton set for every y_0^*. The intersection of singleton sets in (14) can either be a singleton set or an empty set. Since V is concave, the subdifferential $\partial V(x_0)$ is non-empty, and hence it must be singleton. This proves that V is differentiable at x_0. □

Proof of Theorem 3: The proof is similar to that of Theorem 2. Using (10) and Theorem 23.2 in Rockafellar (1970), we obtain that $\phi \in \partial V(x_0)$ if and only if

$$\max_{y_0^* \in Y^*(x_0)} \left[D_x f(x_0, y_0^*) + \sum_{i=1}^{k} \lambda_{i0}^* D_x h_i(x_0, y_0^*)\right] \hat{x} \geq \phi \hat{x} \quad \text{for every } \hat{x}, \tag{91}$$

for every $\lambda_0^* \in \Lambda^*(x_0)$. The left-hand side of (91) is the support function of the compact (but not necessarily convex) set

$$\bigcup_{y_0^* \in Y^*(x_0)} \{D_x f(x_0, y_0^*) + \sum_{i=1}^{k} \lambda_{i0}^* D_x h_i(x_0, y_0^*)\}.$$

Theorem 13.1 in Rockafellar (1970) implies that $\phi \in \partial V(x_0)$ if and only if

$$\phi \in \bigcap_{\lambda_0^* \in \Lambda^*(x_0)} co\left(\bigcup_{y_0^* \in Y^*(x_0)} \{D_x f(x_0, y_0^*) + \sum_{i=1}^{k} \lambda_{i0}^* D_x h_i(x_0, y_0^*)\} \right).$$

□

Proof of Corollary 3: The proof is analogous to that of Corollary 2.

Proof of Proposition 3: We prove first that $\{x_t^*\}$ maximizes the Lagrangian in (19) when the sequence of multipliers is fixed as $\{\lambda_t^*\}$. Consider any sequence $\{x_t\} \in \ell_\infty^n$ such that $x_t \in X$ and $x_0^* = x_0$. Let

$$D_T = \sum_{t=0}^{T} \beta^t \left\{ F(x_t^*, x_{t+1}^*) + \lambda_{t+1}^* h(x_t^*, x_{t+1}^*) - \left[F(x_t, x_{t+1}) + \lambda_{t+1}^* h(x_t, x_{t+1})\right] \right\}. \tag{92}$$

Using the assumption of concavity of F and h_i, we obtain from (92) that

$$\begin{aligned} D_T \geq \sum_{t=0}^{T} \beta^t \{ & \left[D_x F(x_t^*, x_{t+1}^*) + \lambda_{t+1}^* D_x h(x_t^*, x_{t+1}^*)\right] (x_t^* - x_t) \\ & + \left[D_y F(x_t^*, x_{t+1}^*) + \lambda_{t+1}^* D_y h(x_t^*, x_{t+1}^*)\right] \left(x_{t+1}^* - x_{t+1}\right) \}. \end{aligned}$$

Rearranging terms we have

$$\begin{aligned} D_T \ \geq \ \textstyle\sum_{t=0}^{T-1} \beta^t \big\{ \big[D_y F(x_t^*, x_{t+1}^*) + \lambda_{t+1}^* D_y h(x_t^*, x_{t+1}^*) \\ +\beta \ \big[D_x F(x_{t+1}^*, x_{t+2}^*) + \lambda_{t+2}^* D_x h(x_{t+1}^*, x_{t++2}^*) \big] \big] \left(x_{t+1}^* - x_{t+1} \right) \big\} \\ +\beta^T \left[D_y F(x_T^*, x_{T+1}^*) + \lambda_{T+1}^* D_y h(x_T^*, x_{T+1}^*) \right] \left(x_{T+1}^* - x_{T+1} \right). \end{aligned}$$

Using Euler equation (20), we obtain

$$D_T \geq -\beta^{T+1} \left[D_x F(x_{T+1}^*, x_{T+2}^*) + \lambda_{T+2}^* D_x h(x_{T+1}^*, x_{T+2}^*) \right] \left(x_{T+1}^* - x_{T+1} \right). \tag{93}$$

Substituting (92) into (93), we obtain

$$\begin{aligned} \sum_{t=0}^{T} \beta^t \left[F(x_t^*, x_{t+1}^*) + \lambda_{t+1}^* h(x_t^*, x_{t+1}^*) \right] - \sum_{t=0}^{T} \beta^t \left[F(x_t, x_{t+1}) + \lambda_{t+1}^* h(x_t, x_{t+1}) \right] \\ \geq -\beta^{T+1} \left[D_x F(x_{T+1}^*, x_{T+2}^*) + \lambda_{T+2}^* D_x h(x_{T+1}^*, x_{T+2}^*) \right] \left(x_{T+1}^* - x_{T+1} \right) \end{aligned} \tag{94}$$

for every T. Next, we take limits in (94) as T goes to infinity. Since sequences $\{x_t\}$ and $\{x_t^*\}$ are bounded, the transversality condition (22) implies that the RHS is zero in the limit. Complementary slackness conditions imply that the first term on the LHS converges to $\sum_{t=0}^{\infty} \beta^t F(x_t^*, x_{t+1}^*)$. This implies that the second term has a well defined limit and that $\{x_t^*\}$ maximizes the Lagrangian at $\{\lambda_t^*\}$.

The proof that $\{\lambda_t^*\}$ minimizes the Lagrangian in (19) when the sequence of choice variables is fixed as $\{x_t^*\}$ is straightforward and omitted. □

Proof of Proposition 5: The proof is similar to that of Proposition 3. We show first that $\{\mu_t^*, \gamma_t^*\}$ minimizes the Lagrangian in (42) when $\{c_t^*, \lambda_t^*\}$ are fixed. Consider any sequence $\{\mu_t, \gamma_t\}$ such that $\{\mu_t\} \in \ell_\infty^m$ and $\mu_t \geq 0, \gamma_t \geq 0$ and $\mu_0 = \mu$. Let D_T be the difference between date-T partial sums of the Lagrangians for $\{\mu_t^*, \gamma_t^*\}$ and $\{\mu_t, \gamma_t\}$. We have

$$\begin{aligned} D_T = \sum_{t=0}^{T} \beta^t \Bigg\{ \sum_{i=1}^{m} \big[\Delta\mu_{i,t+1} \left(u(c_{i,t}^*) - v_i(y_{i,t}) \right) + \Delta\mu_{i,t} v_i(y_{i,t}) \\ -\lambda_{i,t+1}^* (\Delta\mu_{i,t+1} - \Delta\mu_{i,t}) + (\gamma_{t+1}^* - \gamma_{t+1})(y_{i,t} - c_{i,t}^*) \big] \Bigg\}, \end{aligned}$$

where $\Delta\mu_t = \mu_t^* - \mu_t$. It follows from the Euler equation (43) and complementary slackness that

$$D_T = \beta^{T+1} \sum_{i=1}^{m} \left\{ -[\mu_{i,T+1}^* - \mu_{i,T+1}][v_i(y_{i,T+1}) + \lambda_{i,T+2}^*] - \gamma_{t+1}(y_{i,t} - c_{i,t}^*) \right\}. \tag{95}$$

Since $\gamma_{t+1} \geq 0$ and $c^*_{i,t} \leq y_{i,t}$, it follows from (95) that

$$D_T \leq -\beta^{T+1} \sum_{i=1}^{m} [\mu^*_{i,T+1} - \mu_{i,T+1}][v_i(y_{i,T+1}) + \lambda^*_{i,T+2}]. \tag{96}$$

Since $\mu_{i,T}$ and $\mu^*_{i,T}$ are bounded, the transversality condition (44) implies that the limit on the RHS of (96) is zero. Therefore $\lim_{T\to\infty} D_T \leq 0$.

Next we prove that $\{c^*_t, \lambda^*_t\}$ maximizes the Lagrangian in (42) when $\{\mu^*_t, \gamma^*_t\}$ are fixed. Let

$$\hat{D}_T = \sum_{t=0}^{T} \beta^t \left\{ \sum_{i=1}^{m} \left[\mu^*_{i,t+1}(u(c^*_{i,t}) - u(c_{i,t})) + \gamma^*_{t+1}(c_{i,t} - c^*_{i,t}) \right] \right\}.$$

Using concavity of u, we have

$$\hat{D}_T \geq \sum_{t=0}^{T} \beta^t \left\{ \sum_{i=1}^{m} \left[\mu^*_{i,t+1} u'(c^*_{i,t})(c^*_{i,t} - c_{i,t}) + \gamma^*_{t+1}(c_{i,t} - c^*_{i,t}) \right] \right\}. \tag{97}$$

Using the first-order condition $\gamma^*_{t+1} = \mu^*_{i,t+1} u'(c^*_{i,t})$, it follows from (97) that $\hat{D}_T \geq 0$ and consequently $\lim_{T\to\infty} \hat{D}_T \geq 0$. □

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